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Finitely convergent cutting planes for concave minimization

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Abstract. In 1964 Tuy introduced a new type of cutting plane, the concavity cut, in the context of concave minimization. These cutting planes, which are also known as convexity cuts, intersection cuts and Tuy cuts, have found application in several algorithms, e.g., branch and bound algorithm, conical algorithm and cutting plane algorithm, and also in algorithms for other optimization problems, e.g., reverse convex programming, bilinear programming and Lipschitzian optimization. Up to now, however, it has not been possible to either prove or disprove the finite convergence of a pure cutting plane algorithm for concave minimization based solely on these cutting planes. In the present paper a modification of the concavity cut is proposed that yields deeper cutting planes and ensures the finite convergence of a pure cutting plane algorithm based on these cuts.

Key words: Concave minimization, cutting plane, convexity cut, concavity cut, Tuy cut, finite convergence.

1. Introduction

In this paper we are concerned with the minimization of a concave function $f(x)$ with $f : \mathbb{R}^n \mapsto \mathbb{R}$ over a nonempty polytope P with $P \subset \mathbb{R}^n$. Concave minimization problems of this type have been encountered, for instance, in the context of site selection, inventory management, production and location problems and transportation planning (cf., e.g., Horst and Tuy, 1996).

Most of the difficulties with this type of optimization problem arise because a concave minimization problem may have a very large, even an exponentially large number of local optimal solutions (cf., e.g., Kalantari, 1986), and no local criteria are known that allow us to determine whether a local optimal solution is also a global one or not. Pardalos and Schnitger (1988) showed that even a problem as simple as minimizing a concave quadratic function over a hypercube is $\mathcal{NP}$-hard. However, it is well known that a vertex of the polytope P exists which is a global optimal solution.

One of the most popular algorithms in concave minimization, the *conical algorithm* (a special type of branch and bound algorithm), was first proposed in 1964 in a seminal paper by Tuy. In 1973, however, by counterexample, the algorithm was shown not to be convergent (Zwart, 1973). Since then several modifications of the algorithm have been proposed with the goal of ensuring its (finite) convergence (cf., e.g., Thoai and Tuy, 1980, Horst and Tuy, 1996; Tuy, 1998).

To perform a bounding operation in the conical algorithm Tuy introduced the concept of *concavity cuts*, which in the literature are now also known as convexity cuts, intersection cuts and, in his honor, Tuy cuts. Apart from concave minimization, concavity cuts have also been applied to other types of optimization problems. For instance, for integer programming Balas (1971) proposed a cutting plane algorithm based on concavity cuts and proved its finite convergence. And for the Generalized Lattice Point Problem Sen and Sherali (1985) provided an example of non-convergence of Tuy-type concavity/disjunctive cuts, along with certain information-processing rules for making a cutting plane algorithm convergent. Further examples of problems to which concavity cuts have been applied are bilinear programming (e.g., Konno, 1976; 1981; Sherali and Shetty, 1980; Vaish and Shetty, 1977), reverse-convex programming (e.g., Gurlitz and Jacobsen, 1991; Hillestad and Jacobsen, 1980; and Sen and Sherali, 1987), Lipschitzian optimization (e.g., Bulatov, 1990) and zero-one integer programming (e.g., Young, 1971).

Even though Tuy also outlined a cutting plane algorithm for concave minimization based on concavity cuts (cf. Tuy, 1964, Remark 3), it was Cabot (1974) who was the first to explicitly propose a pure cutting plane algorithm using concavity cuts. Since then several other versions have been suggested, most of them resembling an algorithm recently proposed by Horst and Tuy (1996). It is still unknown, however, whether or not the finite convergence of these algorithms can be ensured solely by concavity cuts.

To be on the safe side some authors have introduced enumerative elements in their cutting plane algorithms. For instance, Konno (1980) considers the maximization of a convex function over a hypercube, which is equivalent to a concave minimization problem. Exploiting the special structure of this problem, he ensures the finite convergence of his cutting plane algorithm by examining and eliminating vertices of the reduced polytope that were also vertices of the original polytope, i.e., that were binary.

For general concave minimization problems the finite convergence of a cutting plane algorithm is often ensured by incorporating facial cuts from time to time (cf., e.g., Tuy and Horst, 1996; and Sherali and Shetty, 1980, in the context of bilinear programming). Facial cuts, which were introduced by Majthay and Whinston (1974), eliminate faces of the reduced polyhedron that are also faces of the original polyhedron. However, to derive a facial cut one first has to identify such a face with using a special procedure.

Our goal in this paper is to modify concavity cuts in a way that on the one hand we obtain deeper cutting planes and on the other finite convergence of a cutting plane algorithm is ensured by the cutting planes themselves. In the next section of the paper we discuss the basic concepts of concavity cuts in the context of concave minimization. In the third and fourth sections we present our modifications of the concavity cut concept. In the fifth section we describe an iterative procedure for deepening the resulting cutting planes. The final section contains some concluding remarks.

2. Concavity Cuts and Cutting Plane Algorithms

In the following we consider the concave minimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & x \in P, \end{array} \tag{2.1}$$

where $f : I\!R^n \mapsto I\!R$ is concave on $I\!R^n$ and $P = \{x \in I\!R^n \mid Ax \leq b\}$ is a polyhedron. For the sake of simplicity we assume that P is bounded and $\dim(P) = n$, i.e., P is a full-dimensional polytope, and that for any real number γ the level set

$$L(\gamma) = \{x \in I\!R^n \mid f(x) \geq \gamma\}$$

is closed and bounded. Note that since $f(x)$ is concave the sets $L(\gamma)$ are convex. Furthermore, we content ourselves with finding an global ε-optimal solution, i.e. a solution $\widehat{x} \in P$ with

$$f(\widehat{x}) \leq f(x) + \varepsilon \quad \forall \ x \in P,$$

where $\varepsilon > 0$ is an arbitrary but predetermined tolerance.

A cutting plane algorithm for concave minimization consists of two alternating phases: search and cut. In the search phase we find a local optimum x_0 and in the cut phase we eliminate x_0 without excluding a solution $x \in P$ with $f(x) < \widehat{f} - \varepsilon$, where $\widehat{f}$ denotes the objective value of the best solution known so far. Before giving the cutting plane algorithm in pseudo code we describe the two phases in more detail.

Using the well-known fact that there exists a vertex of the polytope P which is a global optimum of (2.1), in the search phase we can restrict our search to the vertices of P. In this context we call a vertex x_0 of P a *local optimum* or a *star optimum* if and only if there exists no vertex of P adjacent to x_0 with a smaller value. The procedure for identifying a star optimum is straightforward. Starting at an arbitrary vertex of P we examine its adjacent vertices. Note that special care must be taken to ensure that in case of degeneracy *all* adjacent vertices are enumerated. If there is one with a smaller value we pivot to the adjacent vertex with the smallest value and examine its adjacent vertices. Otherwise we stop. This procedure terminates after a finite number of iterations with a star optimum x_0.

In the cut phase we eliminate x_0 with a cutting plane, e.g., a concavity cut. To this end we assume that x_0 is a nondegenerate vertex of P. Hence there exist exactly n vertices $x_1, x_2, \ldots, x_n$ of P that are adjacent to x_0. Therefore the directions of the edges of P emanating from x_0 can be given w.l.o.g. by $u_i = x_i - x_0$ for $i = 1, \ldots, n$. Note that $u_1, \ldots, u_n$ are linearly independent. With this the cone

$$C(x_0) = x_0 + \text{cone}(u_1, u_2, \ldots, u_n)$$

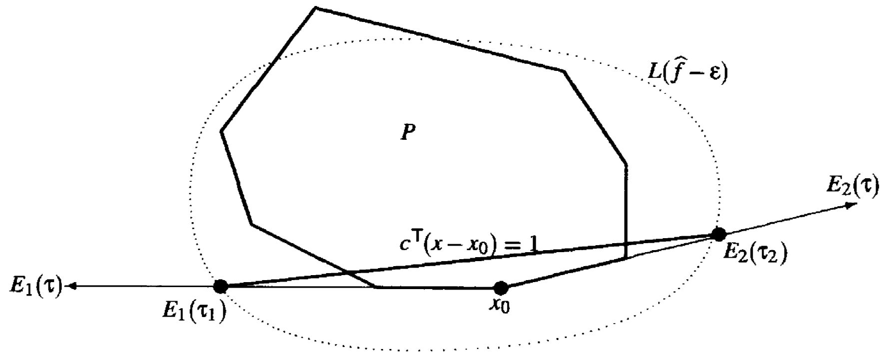

Figure 1. Deriving a concavity cut

is the smallest cone vertexed at x_0 that contains P. This approximation of P allows us to derive easily a cutting plane which eliminates with x_0 only points in $L(\widehat{f}-\varepsilon)$, i.e., it eliminates no point $x \in P$ with $f(x) < \widehat{f} - \varepsilon$. Such a cutting plane is called a *valid cut*. A valid cut can be derived as follows. First we determine the intersection points $E_i(\tau_i)$ of the cone edges $E_i(\tau) = x_0 + \tau u_i$, $\tau \geq 0$, with the boundary of $L(\widehat{f} - \varepsilon)$. Because of $f(x_0) \geq \widehat{f}$ and $f(x_i) \geq \widehat{f}$ we have $\tau_i > 1$ for $i = 1, 2, \ldots, n$. Then we determine the unique hyperplane $c^{\mathsf{T}}(x - x_0) = 1$ that contains these points, i.e.

$$c^{\mathsf{T}} = \left(\frac{1}{\tau_1}, \frac{1}{\tau_2}, \ldots, \frac{1}{\tau_n}\right)^{\mathsf{T}} U^{-1}, \tag{2.2}$$

where $U = (u_1, u_2, \ldots, u_n)$, and y^{T} denotes the transpose of a vector y. By construction

$$C(x_0) \cap \{x \in I\!R^n \mid c^{\mathsf{T}}(x - x_0) \leq 1\}$$

is a simplex that is contained in $L(\widehat{f} - \varepsilon)$. Therefore, and because of $P \subset C(x_0)$, the concavity cut $c^{\mathsf{T}}(x - x_0) \geq 1$ eliminates x_0 but no $x \in P$ with $f(x) < \widehat{f} - \varepsilon$, i.e., it is a valid cut (cf. Figure 1).

To derive a concavity cut we had to assume that the vertex x_0 of P was nondegenerate. If this is not the case, then there might be more than n vertices of P adjacent to x_0, e.g., $x_1, x_2, \ldots, x_r$ with $r \geq n$. By defining $u_i := x_i - x_0$ the cone

$$C(x_0) = x_0 + \mathrm{cone}(u_1, u_2, \ldots, u_r) \tag{2.3}$$

is also the smallest cone vertexed at x_0 that contains P, but $u_1, u_2, \ldots, u_r$ are in case of $r > n$ no longer linearly independent. Hence there may not exist a hyperplane $c^{\mathsf{T}}(x - x_0) = 1$ that passes through all the intersection points $E_i(\tau_i)$. In this case we consider, as proposed by Carvajal-Moreno (1972) and Benson (1999), a basic feasible solution of the system

$$c^{\mathsf{T}}(E(\tau_i) - x_0) \geq 1 \qquad \text{for } i = 1, 2, \ldots, r,$$

which is equivalent to

$$c^{\mathsf{T}} u_i \geq \frac{1}{\tau_i} \qquad \text{for } i = 1, 2, \ldots, r. \tag{2.4}$$

Hence the resulting cutting plane $c^{\mathsf{T}}(x - x_0) \geq 1$ intersects all edges of $C(x_0)$ in $L(\widehat{f} - \varepsilon)$ and contains at least n intersection points $E(\tau_i)$. Obviously $c^{\mathsf{T}}(x - x_0) \geq 1$ eliminates x_0 without excluding a point $x \in P$ with $f(x) < \widehat{f} - \varepsilon$, i.e., it is a valid cut. Other ways to deal with degeneracy in the context of concavity cuts are described, for instance, in Horst and Tuy (1996). Appropriate modifications of the concavity cut concept for cases where the objective function $f(x)$ of (2.1) is quasiconcave or the polytope P is not full dimensional are proposed by Benson (1999).

Based on the concepts described above the structure of a pure cutting plane algorithm for problem (2.1) is as follows.

ALGORITHM I

Choose ε with $\varepsilon > 0$;
Set $\widehat{f} := \infty$, $P_0 := P$ **and** $k := 0$;
While $P_k \neq \emptyset$ **do**
begin
search w.r.t. P_k and $f(x)$ for a star optimum x_{0_k};
if $f(x_{0_k}) < \widehat{f}$**, then set** $\widehat{f} := f(x_{0_k})$ **and** $\widehat{x} := x_{0_k}$;
derive w.r.t. P_k and $f(x)$ a concavity cut $c_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$;
set $P_{k+1} := P_k \cap \{x \in I\!R^n \mid c_k^{\mathsf{T}}(x - x_{0_k}) \geq 1\}$ **and** $k := k + 1$;
end.

In Algorithm I we were not very specific about how to choose a starting point for the search for a star optimum. There are several ways to do this (see, e.g., Horst and Tuy, 1996; Porembski, 1999; or Zwart, 1971). Even though this may have a great impact on the speed of the algorithm, it does not influence its convergence. Therefore we do not go into any more detail about this matter.

If Algorithm I terminates, the best solution known so far, $\widehat{x}$, is an global ε-optimal solution of problem (2.1). However, it is still unknown whether this cutting plane algorithm always terminates after a finite number of iterations or not, i.e. whether we can always eliminate the polytope P with a finite number of concavity cuts? Experiments in the mid-1970s (cf., e.g., Zwart, 1971) showed that the concavity cuts used in a pure cutting plane algorithm such as Algorithm I tend to become more and more shallow, thereby slowing down the search process. Hence it makes sense to formulate a condition for the finite convergence of Algorithm I as follows (see Horst and Tuy, 1996, Theorem V.2).

THEOREM 2.1 *If the sequence* $\{c_k\}$ *is bounded, then Algorithm I is finite.*

Theorem 2.1 states that if there exists a constant α such that we always have $\|c_k\| < \alpha$, where $\|\cdot\|$ denotes the Euclidian norm, then the cutting plane algorithm is finite. Note that $1/\|c_k\|$ is the distance from x_{0_k} to the hyperplane $c_k^{\mathsf{T}}(x - x_{0_k}) = 1$, which is often used as a measure of the depth of the cut.

Horst and Tuy (1996) formulated the condition in Theorem 2.1 in the context of concavity cuts. However, Theorem 2.1 holds for an arbitrary valid cut as long as this cut is of the form $c_k^{\mathsf{T}}(x - x_0) \geq 1$. Even though the condition given in Theorem 2.1 seems to be quite simple, it is usually difficult to satisfy this condition in a pure cutting plane algorithm using concavity cuts.

3. Modifying Concavity Cuts: Basic Concepts

In this section we provide a modification of the concept underlying concavity cuts. Our goal is to ensure for the resulting cuts the fulfillment of a condition which is equivalent to the condition given in Theorem 2.1. As a basis for the following considerations let us have a look at Theorem 2.1 from a different point of view, where to simplify the notation we omit the index k, which stands for the kth iteration of Algorithm I.

Let $c^{\mathsf{T}}(x - x_0) \geq 1$ denote the concavity cut derived w.r.t. the cone $C(x_0) = x_0 + \text{cone}(u_1, u_2, \ldots, u_l)$, where $l = n$ if x_0 is nondegenerate and $l \geq n$ otherwise. By construction x_0 is the only point of P and $C(x_0)$ that is supported by the hyperplane $c^{\mathsf{T}}(x - x_0) = 0$ (cf. Figure 2). Let φ_i with $0 \leq \varphi_i \leq \frac{\pi}{2}$ be chosen such that

$$\cos\varphi_i = \frac{\langle c, u_i \rangle}{\|c\|\|u_i\|} \qquad \text{for } i = 1, 2, \ldots, l, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. Hence φ_i gives us the angle between the normal of the hyperplane $c^{\mathsf{T}}(x - x_0) = 0$ and the ith edge of the cone $C(x_0)$. Furthermore, let, as above, $E_i(\tau_i)$ be the intersection point of the cone edge $E_i(\tau)$ with the boundary of $L(\widehat{f} - \gamma)$ and let $E_i(\bar{\tau}_i)$ denote the point where the hyperplane $c^{\mathsf{T}}(x - x_0) = 1$ passes through the ith edge of $C(x_0)$, i.e. $\bar{\tau}_i = \tau_i$ in case of $l = n$ and $0 < \bar{\tau}_i \leq \tau_i$ in case of $l > n$. Then the distance from $E_i(\bar{\tau}_i)$ to x_0 is $\|\bar{\tau}_i u_i\|$. Hence the distance from x_0 to the point where the cut $c^{\mathsf{T}}(x - x_0) \geq 1$ intersects the ray $x_0 + \lambda c$, $\lambda \geq 0$ is

$$\cos\varphi_i \cdot \|\bar{\tau}_i u_i\| \tag{3.6}$$

(cf. Figure 2). Note that (3.6) gives us the depth of the cut $c^{\mathsf{T}}(x - x_0) \geq 1$. Hence we have

$$\frac{1}{\|c\|} = \cos\varphi_i \cdot \|\bar{\tau}_i u_i\| \qquad \text{for } i = 1, 2, \ldots, l. \tag{3.7}$$

This observation can be used to obtain an equivalent formulation of Theorem 2.1.

THEOREM 3.1 *Let $c_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ be the concavity cut that was derived in the kth iteration of Algorithm I, and let φ_{i_k} for $i_k = 1, 2, \ldots, l_k$ be the corresponding*

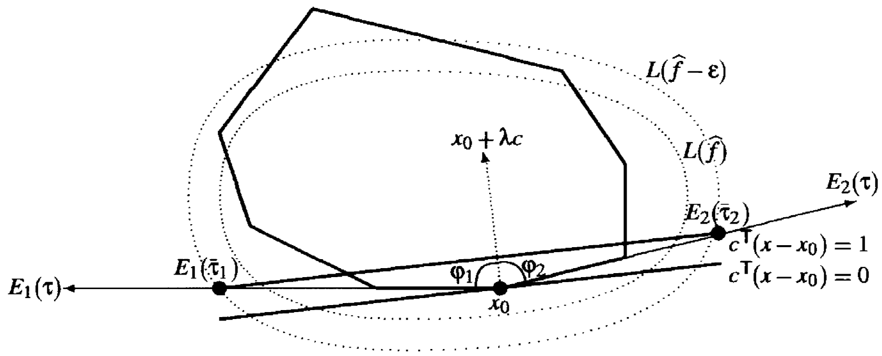

Figure 2. Definition of φ_i

angles that are determined according to (3.5). $\{c_k\}$ *is bounded if and only if there exists a constant* κ *with* $\kappa > 0$ *such that for* $k = 1, 2, \ldots$

$$\kappa \le \min_{i_k=1}^{l_k} \cos\varphi_{i_k}, \tag{3.8}$$

holds.

Proof. Let us first suppose that there exists a constant α such that $\|c_k\| < \alpha$. Then we have

$$\frac{1}{\alpha} < \frac{1}{\|c_k\|} = \cos\varphi_{i_k} \cdot \|\bar{\tau}_{i_k} u_{i_k}\| \qquad \text{for } \; i_k = 1, 2, \ldots, l_k.$$

Let f_{opt} denote the objective value of the global optimal solution of the concave minimization problem (2.1) and let Δ be the diameter of the level set $L(f_{\text{opt}} - \varepsilon)$, which by assumption is bounded. Therefore, we have $1/\alpha < \cos\varphi_{i_k} \cdot \Delta$, which is equivalent to

$$\frac{1}{\alpha \cdot \Delta} < \cos\varphi_{i_k}$$

for $i_k = 1, 2, \ldots, l_k$ and $k = 1, 2, \ldots,$ and hence there exists a constant κ with $\kappa > 0$ such that condition (3.8) is satisfied.

Now we have to prove the converse direction. To do this let us suppose that there exists a constant $\kappa > 0$ such that condition (3.8) holds. Since $f(x)$ is continuous and the level sets $L(\gamma)$ are compact by assumption there exists a constant $\delta > 0$ such that for all $\gamma \ge f_{\text{opt}}$ with $L(\gamma) \ne \emptyset$ the distance from the boundary of $L(\gamma)$ to the boundary of $L(\gamma - \varepsilon)$ is greater than δ. Furthermore, by construction of the concavity cut $c_k^{\mathrm{T}}(x - x_{0_k}) \ge 1$ the equation $c_k^{\mathrm{T}}(E_{i_k}(\tau_{i_k}) - x_0) = 1$ holds for at least one index $i_k \in \{1, 2, \ldots, l_k\}$, i.e. $\tau_{i_k} = \bar{\tau}_{i_k}$. Because of $E_{i_k}(\tau_{i_k}) \in \text{bd}(L(\widehat{f} - \varepsilon))$ and $x_{0_k} \in L(\widehat{f})$ we therefore have

$$\delta < \|E_{i_k}(\tau_{i_k}) - x_{0_k}\| \;=\; \|\tau_{i_k} u_{i_k}\|.$$

Applying assumption (3.8), with this and (3.7) we get

$$\frac{1}{\|c_k\|} = \cos\varphi_{i_k} \cdot \|\tau_{i_k} u_{i_k}\| > \kappa \cdot \delta \;\; > \;\; 0.$$

Hence we have $\|c_k\| < 1/(\kappa \cdot \delta)$, which verifies Theorem 3.1. □

Similar to the condition given in Theorem 2.1, the condition in Theorem 3.1 is difficult to satisfy. However, Theorem 3.1 gives us an idea about how we can ensure the finite convergence of a cutting plane algorithm by modifying the concept of concavity cuts. Before we go into more detail it will be helpful for what follows to generalize the cone $C(x_0)$ that we used above to derive a concavity cut.

DEFINITION 3.1 *Let* $x_0 \in \mathbb{R}^n$ *such that* $x_0 \notin \operatorname{int}(P)$, *where* P *is a full-dimensional polytope. Then* $C(x_0)$ *denotes the smallest cone of the form* $C(x_0) = x_0 + \operatorname{cone}(u_1, u_2, \ldots, u_s)$ *with* $s \geq n$ *that contains* P.

If x_0 is a vertex of the polytope P, then the cone defined in Definition 3.1 is identical with the cone used to derive a concavity cut.

Let $c^{\mathsf{T}}(x - x_0) \geq 1$ be the concavity cut derived w.r.t. the vertex x_0 of the polytope P, where $x_0 \in L(\widehat{f})$. If this cut is very shallow, then $\min_{i=1}^{l} \cos\varphi_i$ is also very small. The idea is now to increase $\min_{i=1}^{l} \cos\varphi_i$ by pulling the base of the cone $C(x_0)$ in the direction $-c$, i.e. we consider a cone $C(x_0')$ with $x_0' = x_0 - \lambda_0 c$, where $\lambda_0 > 0$. Here we choose λ_0 such that x_0' lies on the boundary of $L(\widehat{f} - \varepsilon)$ (cf. Figure 3). Note that by assumption $x_0 \in L(\widehat{f})$.

An algorithm for obtaining an explicit representation of the resulting cone $C(x_0')$ is given in the fourth section. For what follows let us suppose that $C(x_0')$ is of the form

$$C(x_0') = x_0' + \operatorname{cone}(u_1', u_2', \ldots, u_s').$$

Since P is a polytope we can assume w.l.o.g. that there exist vertices $x_1', x_2', \ldots, x_s'$ of P such that $u_i' = x_i' - x_0'$ for $i = 1, 2, \ldots, s$. Furthermore, we can assume w.l.o.g. $f(x_i') \geq \widehat{f}$, because otherwise x_i' is a solution of problem (2.1) with a smaller objective value than the best solution known so far. It holds:

THEOREM 3.2 *Let the cone* $C(x_{0_k}') = x_{0_k}' + \operatorname{cone}(u_{1_k}', u_{2_k}', \ldots, u_{s_k}')$ *be constructed in the* k*th iteration of a cutting plane algorithm in the way described above, where* $x_{0_k}' \in \operatorname{bd}(L(\widehat{f} - \varepsilon))$. *Then there exists a constant* $\kappa' > 0$ *such that*

$$\kappa' \leq \cos\varphi_{i_k}' = \frac{\langle c_k, u_{i_k}' \rangle}{\|c_k\| \|u_{i_k}'\|} \quad \textit{for } i_k = 1, 2, \ldots, s_k \textit{ and } k = 1, 2, \ldots, \tag{3.9}$$

where $c_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ *denotes the corresponding concavity cut.*

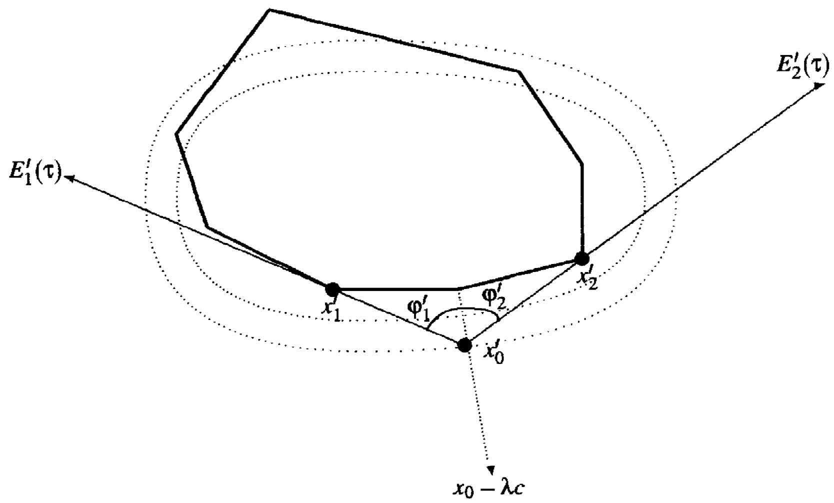

Figure 3. The cone $C(x_0')$

Proof. Consider the two-dimensional simplex spanned by x_{0_k}, x'_{0_k} and $E'_{i_k}(\widehat{\tau}_{i_k})$, where $E'_{i_k}(\widehat{\tau}_{i_k})$ denotes the point where the hyperplane $c_k^{\mathsf{T}}(x - x_{0_k}) = 0$ passes through the edge $E'_{i_k}(\tau) = x_{0_k} + \tau u'_{i_k}$, $\tau \geq 0$, of $C(x'_{0_k})$ (cf. Figure 4). This simplex is contained in $L(\widehat{f} - \varepsilon)$ and it holds

$$\|x_{0_k} - x'_{0_k}\| = \cos \varphi'_{i_k} \cdot \|E'_{i_k}(\widehat{\tau}_{i_k}) - x'_{0_k}\|. \tag{3.10}$$

Since $x_{0_k} \in L(\widehat{f})$ and $x'_{0_k} \in \mathrm{bd}(L(\widehat{f} - \varepsilon))$ it holds $\|x_{0_k} - x'_{0_k}\| > \delta$, where δ is defined as in the proof of Theorem 3.1. Furthermore, we obviously have $\|E'_{i_k}(\widehat{\tau}_{i_k}) - x'_{i_0}\| \leq \Delta$, where Δ is also defined in the same way as in the previous proof. With (3.10) we therefore obtain

$$\frac{\delta}{\Delta} \leq \cos \varphi'_{i_k},$$

which proves Theorem 3.2. □

Now that we have the cone $C(x_0')$ we can use it in a way similar to how we used the cone $C(x_0)$ to derive a cutting plane. To this end we determine the intersection points $E_i'(\tau_i')$ of the edges $E_i'(\tau) = x_0' + \tau u_i'$, $\tau \geq 0$, of $C(x_0')$ with the boundary of $L(\widehat{f} - \varepsilon)$. However, to ensure that the resulting cutting plane has a certain depth, we have to make some modifications to the concavity cut concept.

Again let $c^{\mathsf{T}}(x - x_0) \geq 1$ denote the concavity cut derived w.r.t. x_0 and consider the hyperplane $c^{\mathsf{T}}(x - x_0) = 0$. This hyperplane supports the polytope P at x_0 and intersects the edges of the cone $C(x_0')$ in the interior of $L(\widehat{f} - \varepsilon)$. Let, as in the proof of Theorem 3.2, $E_i'(\widehat{\tau}_i)$, $i = 1, 2, \ldots, s$ denote these intersection points. By assumption we have $u_i' = x_i' - x_0$ for $i = 1, 2, \ldots, s$, where x_i' is a vertex of P

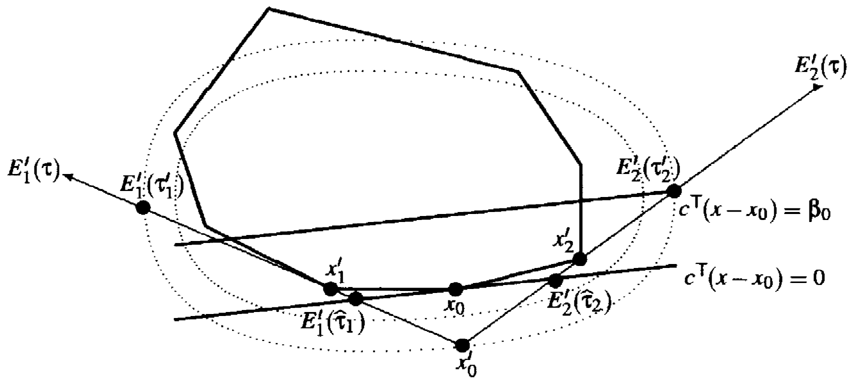

Figure 4. The cut $c^{\mathsf{T}}(x-x_0) \geq \beta_0$

with $x_i' \in L(\widehat{f})$. Obviously for these vertices holds

$$x_i \in \operatorname{conv}\Big(E_i'(\widehat{\tau}_i),\, E_i'(\tau_i')\Big).$$

Since $x_i' \in L(\widehat{f})$ and $E_i'(\tau_i') \in \operatorname{bd}(L(\widehat{f}-\varepsilon))$, we therefore have

$$\|E_i'(\tau_i') - E_i'(\widehat{\tau}_i)\| \;\geq\; \delta \qquad \text{for } i = 1, 2, \ldots, s. \tag{3.11}$$

We can push the hyperplane $c^{\mathsf{T}}(x-x_0) = 0$ forward, i.e. we can increase β until $c^{\mathsf{T}}(x-x_0) = \beta$ passes through at least one of the points $E_1'(\tau_1'), E_2'(\tau_2'), \ldots, E_s'(\tau_s')$ for the first time (cf. Figure 4). Let $c^{\mathsf{T}}(x-x_0) = \beta_0$ denote the resulting hyperplane and let, to be congruent with the notation in Figure 4, w.l.o.g. $c^{\mathsf{T}}(E_2'(\tau_2')-x_0) = \beta_0$. Then $c^{\mathsf{T}}(x-x_0) \geq \beta_0$ is a valid cut, i.e. it eliminates x_0 without excluding a point $x \in P$ with $f(x) < \widehat{f} - \varepsilon$. The depth of this cut is $\beta_0/\|c\|$, where because of Theorem 3.2 and (3.11) holds

$$\frac{\beta_0}{\|c\|} = \cos\varphi_2' \cdot \|E_2'(\tau_2') - E_2'(\widehat{\tau}_2)\| \geq \kappa' \cdot \delta. \tag{3.12}$$

Note that the constants κ' and δ are independent of the respective cut. Hence by the procedure just described we can ensure that the resulting cut $c^{\mathsf{T}}(x-x_0) \geq \beta_0$ has at least a certain depth. Therefore, when in Algorithm I we replace the concavity cut with these cuts, the resulting algorithm is finite. Furthermore, by comparing the corresponding concavity cut and the cut $c^{\mathsf{T}}(x-x_0) \geq \beta_0$ we get the following hierarchy.

THEOREM 3.3 *For the cut $c^{\mathsf{T}}(x-x_0) \geq \beta_0$ it holds $\beta_0 \geq 1$. If $f(x)$ is strictly concave, then we even have $\beta_0 > 1$.*

Proof. Let us denote by $\mathcal{I}$ the intersection of the level set $L(\widehat{f}-\varepsilon)$ and the hyperplane $c^{\mathsf{T}}(x-x_0) = 1$, i.e.

$$\mathcal{I} := L(\widehat{f}-\varepsilon) \cap \{x \in \mathbb{R}^n \mid c^{\mathsf{T}}(x-x_0) = 1\}.$$

The idea of the proof is to show that each edge $E_i'(\tau)$ of the cone $C(x_0')$ intersects $\mathcal{I}$ in its relative interior. If this is the case, then it holds

$$c^{\mathsf{T}}(E_i'(\tau_i') - x_0) \geq 1 \quad \text{for } i = 1, 2, \ldots, s, \tag{3.13}$$

where, as above, $E_i'(\tau_i')$ denotes the intersection point of the edge $E_i'(\tau)$ and $\mathrm{bd}(L(\widehat{f} - \varepsilon))$. If $f(x)$ is strictly concave, then we even have

$$c^{\mathsf{T}}(E_i'(\tau_i') - x_0) > 1 \quad \text{for } i = 1, 2, \ldots, s. \tag{3.14}$$

The inequalities (3.13) and (3.14) follow from the fact that in the first case $f(x) \geq \widehat{f} - \varepsilon$ and in the second case $f(x) > \widehat{f} - \varepsilon$ for all $x \in \mathrm{int}_{\mathrm{rel}}(\mathcal{I})$ holds, where $\mathrm{int}_{\mathrm{rel}}(\cdot)$ denotes the relative interior, which means that the intersection points of the edges $E_i'(\tau)$ with the boundary of $L(\widehat{f} - \varepsilon)$ have to lie in the closed and open half-space, respectively, defined by the concavity cut. Since $c^{\mathsf{T}}(x - x_0) = \beta_0$ contains at least one of these intersection points, we can immediately see that we have $\beta_0 \geq 1$ and $\beta_0 > 1$, respectively.

Now let us consider the edge $E_i'(\tau)$ and let us prove that it intersects $\mathcal{I}$ in its relative interior. By construction each edge $E_i'(\tau)$ of the cone $C(x_0')$ contains at least one vertex of the polytope P. As above, let x_i' be the respective vertex lying on $E_i'(\tau)$, where we assume w.l.o.g. $x_i' \in L(\widehat{f})$. Furthermore, let $E_i'(\bar{\tau}_i')$ denote the intersection point of the edge $E_i'(\tau)$ and the hyperplane $c^{\mathsf{T}}(x - x_0) = 1$. We now have to consider two cases.

Case 1. Suppose that it holds $c^{\mathsf{T}}(x_i' - x_0) \geq 1$. Because of $c^{\mathsf{T}}(x_0' - x_0) < 1$ we therefore have

$$E_i'(\bar{\tau}_i') \in \{(1 - \lambda)x_0' + \lambda x_i' \mid 0 < \lambda \leq 1\}.$$

It follows from $x_0' \in L(\widehat{f} - \varepsilon)$ and $x_i' \in L(\widehat{f})$, $L(\widehat{f}) \subset \mathrm{int}(L(\widehat{f} - \varepsilon))$ and the convexity of $L(\widehat{f} - \varepsilon)$ that $E_i'(\bar{\tau}_i') \in \mathrm{int}(L(\widehat{f} - \varepsilon))$. Hence in this case we have $E_i'(\bar{\tau}_i') \in \mathrm{int}_{\mathrm{rel}}(\mathcal{I})$.

Case 2. Suppose that it holds $c^{\mathsf{T}}(x_i' - x_0) < 1$. Since $x_0' \notin C(x_0)$ and $x_i' \in P \subset C(x_0)$, the edge $E_i'(\tau)$ intersects the boundary of the cone $C(x_0)$ at a point $E_i'(\widetilde{\tau}_i')$ with $c^{\mathsf{T}}(E_i'(\widetilde{\tau}_i') - x_0) < 1$. It is not difficult to verify that because of the special construction of the cone $C(x_0')$ each point $E_i'(\tau)$ with $\tau > \widetilde{\tau}_i'$ is contained in the interior of the cone $C(x_0)$. Because of $c^{\mathsf{T}}(E_i'(\widetilde{\tau}_i') - x_0) < 1$ we have $\bar{\tau}_i' > \widetilde{\tau}_i'$. Since

$$C(x_0) \cap \{x \in \mathbb{R}^n \mid c^{\mathsf{T}}(x - x_0) = 1\} \subset \mathcal{I}$$

we therefore have $E_i'(\bar{\tau}_i') \in \mathrm{int}_{\mathrm{rel}}(\mathcal{I})$, i.e. in this case too the edge $E_i'(\tau)$ intersects the hyperplane $c^{\mathsf{T}}(x - x_0) = 1$ in the interior of $L(\widehat{f} - \varepsilon)$. Together with the considerations above this proves the theorem. □

In general, even if $f(x)$ is not strictly concave, the cut $c^{\mathsf{T}}(x - x_0) \geq \beta_0$ is not only equivalent to but dominates the corresponding concavity cut, i.e., we have

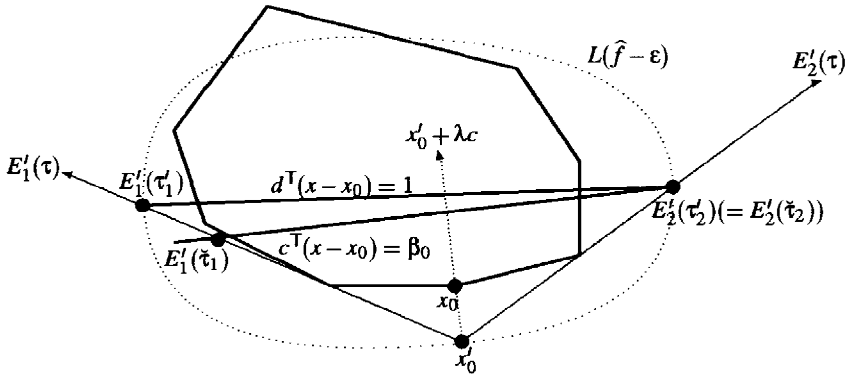

Figure 5. The cut $d^{\mathsf{T}}(x - x_0) \geq 1$

$\beta_0 > 1$. However, in most cases it is not the deepest cut possible that can be derived w.r.t. the cone $C(x_0')$. Obviously this is also the case for the cut indicated in Figure 4. To get a deeper cut we determine the intersection points of the cone edges of $C(x_0')$ with the hyperplane $c^{\mathsf{T}}(x - x_0) = \beta_0$. Let $E_1'(\check{\tau}_1), E_2'(\check{\tau}_2), \ldots, E_s'(\check{\tau}_s)$ be these points. Then we solve the linear program

$$\begin{array}{lll} \text{minimize} & c^{\mathsf{T}}d & \\ \text{s.t.} & d^{\mathsf{T}}(E_i'(\tau_i') - x_0) \geq 1 & \text{for } i = 1, 2, \ldots, s, \\ & d^{\mathsf{T}}(E_i'(\check{\tau}_i) - x_0) \leq 1 & \text{for } i = 1, 2, \ldots, s, \end{array} \tag{3.15}$$

i.e., we determine a valid cut $d^{\mathsf{T}}(x - x_0) \geq 1$ that passes through the edges of $C(x_0')$ between $E_i'(\check{\tau}_i)$ and $E_i'(\tau_i')$ and thereby maximizes the distance from x_0 to its intersection point with the ray $x_0 + \lambda c$, $\lambda \geq 0$. Note that $d = \frac{1}{\beta_0}c$ is a feasible solution of (3.15) and that the linear program (3.15) is bounded since $L(\widehat{f} - \varepsilon)$ is bounded by assumption. By construction the resulting cut $d^{\mathsf{T}}(x - x_0) \geq 1$ dominates or is at least equivalent to the cut $c^{\mathsf{T}}(x - x_0) \geq \beta_0$ and we therefore have

$$\frac{1}{\|d\|} \geq \frac{\beta_0}{\|c\|} \geq \kappa' \cdot \delta.$$

Such a cutting plane is indicated in Figure 5. Comparing this cut with the corresponding concavity cut indicated in Figure 1, we can see that both cuts exclude x_0, but that the cut in Figure 5 eliminates a much larger portion of the polytope P than the concavity cut.

REMARK 3.1 *Since our main interest in the context of cutting plane algorithms is to derive cuts which are as deep as possible, we have focused our attention on concavity cuts. However, each cut $\widetilde{c}^{\mathsf{T}}(x - x_0) \geq 1$ that for $i = 1, 2, \ldots, s$ intersects the ray $E_i(\tau)$ of the 'concavity cut cone' $C(x_0)$ at a point contained in* $\text{conv}(x_0, E_i(\tau_i))$ *is obviously also a valid cut since it is dominated by the concavity* *cut. In the literature such a cut is known as a γ-valid cut, where in our case we have $\gamma = \widehat{f} - \varepsilon$ (see, e.g., Horst and Tuy, 1996). It is not hard to verify that as long as the hyperplane $\widetilde{c}^{\mathsf{T}}(x - x_0) = 1$ contains at least one intersection point of a ray of $C(x_0)$ with $\mathrm{bd}(L(\widehat{f} - \varepsilon))$, all concepts described above also hold true for the γ-valid cut $\widetilde{c}^{\mathsf{T}}(x - x_0) \geq 1$.*

But why consider such a cutting plane? One reason is that it may sometimes be useful to derive a cone $C(x_0')$ by pulling the base of the cone $C(x_0)$ in the direction $-\widetilde{c}$ instead of the direction $-c$, where $\widetilde{c}^{\mathsf{T}}(x - x_0) \geq 1$ is a suitable γ-valid cut and $c^{\mathsf{T}}(x - x_0) \geq 1$ the corresponding concavity cut. Hence we get a whole connected area on the boundary of the level set $L(\widehat{f} - \varepsilon)$ from which we can choose the base x_0' of the cone $C(x_0')$. However, the shallower a concavity cut the larger the angles φ_i determined according to (3.5), and as a result the smaller this area turns out to be.

In spite of Remark 3.1, as above, for the sake of simplicity we will concentrate in the subsequent text on concavity cuts. However, the reader should bear in mind that the considerations also hold true for γ-valid cuts satisfying the conditions stated in Remark 3.1.

4. Cone Adaptation

4.1. THE BASIC PROCEDURE

In the previous section we described a method for deriving cutting planes with a certain depth. To derive these cutting planes we need a representation of the cone $C(x_0')$ of the form

$$C(x_0') = x_0' + \mathrm{cone}(u_1', u_2', \ldots, u_s'). \tag{4.16}$$

We can obtain such a representation in three steps. In the first step we consider the 'concavity cut cone'

$$C(x_0) = x_0 + \mathrm{cone}(u_1, u_2, \ldots, u_l) \tag{4.17}$$

and derive from it the *base cone*

$$\widehat{C}(x_0') = x_0' + \mathrm{cone}(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_l). \tag{4.18}$$

Note that we always have $n \leq l \leq s$. The base cone is vertexed at x_0' and its extreme rays touch the boundary of P but contain no interior points of P. This cone does not necessarily contain the polytope P, but if it does not it can be used to construct such a cone

$$\widetilde{C}(x_0') = x_0' + \mathrm{cone}(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_t) \tag{4.19}$$

with $t \geq s$ (step 2). Since the representation of the cone $\widetilde{C}(x_0')$ might not be minimal, we have to identify among $\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_t$ the extreme directions of $\widetilde{C}(x_0')$, say $\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_s$ (step 3). By setting $u'_i := \widetilde{u}_i$ for $i = 1, 2, \ldots, s$ we obtain a representation of the cone $C(x'_0)$ of the form (4.16).

In the following subsection we discuss the first step in deriving the cone $C(x'_0)$, the construction of the base cone $\widehat{C}(x'_0)$.

4.2. DERIVING THE BASE CONE

The base cone $\widehat{C}(x'_0)$ is derived successively from the concavity cut cone $C(x_0)$. The basic concept is the following. We initially define

$$x_0(\lambda) := x_0 - \lambda c \quad \text{and} \quad u_i^{(0)}(\lambda) := x_i^{(0)} - x_0(\lambda) \quad \text{for } i = 1, 2, \ldots, l, \tag{4.20}$$

where $x_1^{(0)}, x_2^{(0)}, \ldots, x_l^{(0)}$ are the neighboring vertices of x_0 in P, and consider the cone

$$\widehat{C}^{(0)}(x_0(\lambda)) = x_0(\lambda) + \operatorname{cone}\Big(u_1^{(0)}(\lambda), u_2^{(0)}(\lambda), \ldots, u_l^{(0)}(\lambda)\Big). \tag{4.21}$$

Let us denote by $E_{i,\lambda}^{(0)}(\tau) = x_0(\lambda) + \tau u_i^{(0)}(\lambda)$ the ith extreme ray of $\widehat{C}^{(0)}(x_0(\lambda))$ and note that $C(x_0) = \widehat{C}^{(0)}(x_0(0))$. We now increase λ, i.e., we pull x_0 in the direction $-c$ and use $x_1^{(0)}, x_2^{(0)}, \ldots, x_l^{(0)}$ as 'hinges' for the extreme rays of the cone, until we either have $x_0(\lambda) \in \operatorname{bd}(L(\widehat{f} - \varepsilon))$ or at least one extreme ray starts to enter the interior of P. This is illustrated in Figure 6 for the two-dimensional case, where we can increase λ up to λ_2 without entering the interior of P.

Suppose $\lambda^{(1)}$ is the smallest value such that for $\lambda > \lambda^{(1)}$ an extreme ray of $\widehat{C}^{(0)}(x_0(\lambda))$, say $E_{1,\lambda}^{(0)}(\tau)$, enters the interior of P, and suppose that we have $\lambda^{(1)} < \lambda_0$, where λ_0 is determined by $x_0(\lambda_0) \in \operatorname{bd}(L(\widehat{f} - \varepsilon))$. Then there exists a $x_1^{(0)}$-containing facet of P that not only is touched by the extreme ray $E_{1,\lambda^{(1)}}^{(0)}(\tau)$ at some point but also contains part of the ray. $E_{1,\lambda^{(1)}}^{(0)}(\tau)$ enters this facet for $\tau = 1$ at $x_1^{(0)}$ and leaves it for some $\tau_1^{(1)} > 1$ (cf. $E_{1,\lambda_2}^{(0)}(\tau)$ in Figure 6). The unique point $x_1^{(1)} := E_{1,\lambda^{(1)}}^{(0)}(\tau_1^{(1)})$ where the extreme ray $E_{1,\lambda^{(1)}}^{(0)}(\tau)$ leaves this facet is then used as a new hinge for the ray, i.e., we update the direction of the extreme ray for $\lambda > \lambda^{(1)}$ by setting $u_1^{(1)}(\lambda) := x_1^{(1)} - x_0(\lambda)$. This updated direction ensures that the ray does not enter the interior of P when λ is further increased. We define $u_i^{(1)} := u_i^{(0)}$ for $i = 2, 3, \ldots, l$ and then consider for $\lambda > \lambda^{(1)}$ the modified cone

$$\widehat{C}^{(1)}(x_0(\lambda)) = x_0(\lambda) + \operatorname{cone}\Big(u_1^{(1)}(\lambda), u_2^{(1)}(\lambda), \ldots, u_l^{(1)}(\lambda)\Big).$$

This is also illustrated in Figure 6, where $\lambda^{(1)} = \lambda_2$.

Next we further increase λ until either $\lambda = \lambda_0$ or an extreme ray of the modified cone $\widehat{C}^{(1)}(x_0(\lambda))$ starts to enter the interior of P. In the latter case we update the direction of the respective extreme ray as described above, and so on. The process terminates when $\lambda^{(k-1)} < \lambda_0 \leq \lambda^{(k)}$, where $k \geq 1$ and $\lambda^{(0)} := 0$. Then we set $\widehat{C}(x'_0) := \widehat{C}^{(k-1)}(x_0(\lambda_0))$. It is not hard to prove the following.

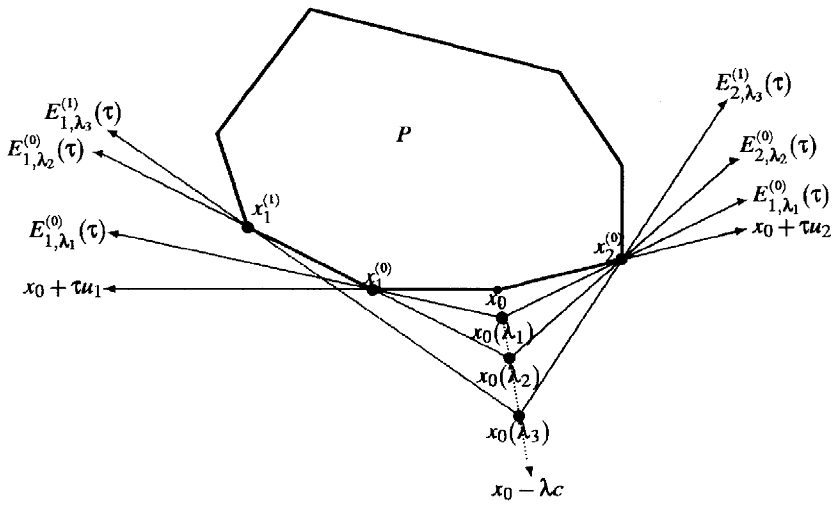

Figure 6. Base cones for different values of λ

PROPOSITION 4.1 *The procedure described above terminates after a finite number of iterations with a cone* $\widehat{C}(x_0') = x_0' + \text{cone}(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_l)$ *with* $x_0' \in \text{bd}(L(\widehat{f} - \varepsilon))$*, for which each cone edge* $\widetilde{E}_i(\tau) = x_0' + \tau\widetilde{u}_i$, $\tau \geq 0$*, touches the polytope* P *at its boundary and contains no interior points of* P.

Finally, let us briefly outline how to determine the values $\lambda^{(k)}$ in the procedure described above.

REMARK 4.1 *If the extreme ray* $E^{(k-1)}_{i,\lambda}(\tau) = x_0(\lambda) + \tau u_i^{(k-1)}$, $\tau \geq 0$*, enters the interior of* P *as* λ *increases, then it intersects the boundary of* P *the first time at a* $x_i^{(k-1)}$*-containing facet of* P *for some* $\tau > 1$*, where* $x_i^{(k-1)}$ *is the incumbent hinge of* $E^{(k-1)}_{i,\lambda}(\tau)$*. Based on this observation, for an extreme ray* $E^{(k-1)}_{i,\lambda}(\tau)$ *we can identify that inequality of the system* $Ax \leq b$ *which describes the facet of* P *intersected by* $E^{(k-1)}_{i,\lambda}(\tau)$*. For this inequality, say* $a_i^{\mathsf{T}}x \leq \beta_i$, $a_i^{\mathsf{T}}x_0(\lambda) < \beta_i$ *holds for* $0 \leq \lambda \leq \lambda^{(k-1)}$ *and* $a_i^{\mathsf{T}}x_i^{(k-1)} = \beta_i$*. Then* $\lambda_i^{(k)}$ *with* $a_i^{\mathsf{T}}(x_0(\lambda^{(k)}) + \tau u_i^{(k-1)}(\lambda^{(k)})) = \beta_i$ *is the smallest value of* λ *for which the extreme ray* $E^{(k-1)}_{i,\lambda}(\tau)$ *starts to enter the interior of* P*. Hence the minimum of the respective* $\lambda_i^{(k)}$ *gives us* $\lambda^{(k)}$.

4.3. CONSTRUCTING A P-CONTAINING CONE

Even though in the two-dimensional case the base cone $\widehat{C}(x_0')$ is always the smallest P-containing cone vertexed at x_0', i.e., $C(x_0') = \widehat{C}(x_0')$, this is not necessarily true when $\dim(P) > 2$. This is illustrated in Figure 7, where the base cone $\widehat{C}(x_0')$ contains only a part of P. For instance, the vertex $\widehat{x}$ of P lies above the facet of $\widehat{C}(x_0')$ spanned by the cone edges $\widetilde{E}_1(\tau)$ and $\widetilde{E}_2(\tau)$.

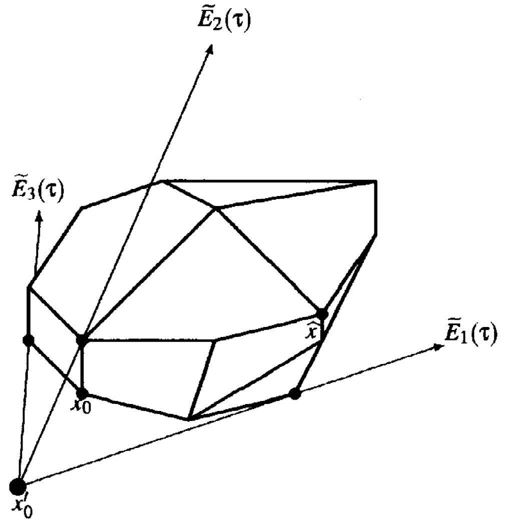

Figure 7. The base cone $\widehat{C}(x_0')$

Therefore, after we have derived the n-dimensional base cone

$$\widehat{C}(x_0') = x_0' + \text{cone}(\widetilde{u}_1, \widetilde{u}_2, \dots, \widetilde{u}_l)$$

we have to determine whether it contains the complete polytope P or not, i.e., $l = t$ or $l < t$ in (4.18) and (4.19). If not, then we have to determine the remaining $t - l$ directions. These steps can be done as follows.

First, we set $\widetilde{C}^{(0)}(x_0') := \widehat{C}(x_0')$ and $k := 0$. Let us now consider the cone $\widetilde{C}^{(k)}(x_0')$. For each facet $\mathcal{G}_i$ of $\widetilde{C}^{(k)}(x_0')$ we determine a hyperplane $g_i^{\mathsf{T}}x \le \theta_i$ such that $\mathcal{G}_i = \{x \in \widetilde{C}^{(k)}(x_0') \mid g_i^{\mathsf{T}}x = \theta_i\}$ and

$$\widetilde{C}^{(k)}(x_0') \subset \{x \in I\!R \mid g_i^{\mathsf{T}}x \le \theta_i\},$$

where we omit, to simplify the notation, the index k whenever possible. Because of the special structure of the cone $\widetilde{C}^{(k)}(x_0')$ such an inequality can be easily determined. With the help of $g_i^{\mathsf{T}}x \le \theta_i$ we can now determine whether there exists a vertex of P that lies ‘above’ the facet $\mathcal{G}_i$ of $\widetilde{C}^{(k)}(x_0')$. For this purpose we solve the linear program

$$\max\{g_i^{\mathsf{T}}x \mid x \in P\}. \tag{4.22}$$

Since by construction each ray $\widetilde{E}_i(\tau) := x_i' + \tau\widetilde{u}_i$, $\tau \ge 0$, of $\widetilde{C}^{(k)}(x_0')$ contains at least one boundary point of P, for an optimal solution $\widehat{x}_i$ of (4.22) it holds $g_i^{\mathsf{T}}\widehat{x}_i \ge \theta_i$. We now have to distinguish between two cases:

Case 1: Suppose it holds $g_i^{\mathsf{T}}\widehat{x}_i = \theta_i$. Then P lies in the half-space defined by $g_i^{\mathsf{T}}x \le \theta_i$, i.e., there exists no vertex of P that lies above the facet $\mathcal{G}_i$ of $\widetilde{C}^{(k)}(x_0')$. In Figure 7 this is the case, for instance, for the facet of $\widehat{C}(x_0')$ $(= \widetilde{C}^{(0)}(x_0'))$ that is spanned by the edges $\widetilde{E}_1(\tau)$ and $\widetilde{E}_3(\tau)$.

Case 2: Suppose we have $g_i^{\mathsf{T}}\widehat{x}_i > \theta_i$, i.e., $P \not\subseteq \widetilde{C}^{(k)}(x_0')$. In Figure 7 this is the case, for instance, for the facet of $\widehat{C}(x_0')$ $(= \widetilde{C}^{(0)}(x_0'))$ that is defined by the edges $\widetilde{E}_1(\tau)$ and $\widetilde{E}_2(\tau)$. In this case we have to enlarge the cone $\widetilde{C}^{(k)}(x_0')$. We do this by adding another direction $\widetilde{u}_{l+k+1}$ to the directions of $\widetilde{C}^{(k)}(x_0')$, where $x_0' + \tau\widetilde{u}_{l+k+1}$, $\tau \geq 0$, contains a vertex of P but no interior points of P, i.e. we define the cone $\widetilde{C}^{(k+1)}(x_0') = x_0' + \mathrm{cone}(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_{l+k+1})$. We set $k := k+1$ and repeat the procedure described above for the enlarged cone.

The procedure terminates when Case 1 for all facets of the cone $\widetilde{C}^{(k)}(x_0')$ holds. Since P has only a finite number of vertices, this condition is always fulfilled after a finite number of iterations.

However, before we can prove that we have obtained with this approach the cone $\widetilde{C}(x_0')$ in (4.19), we have to discuss how to determine an appropriate direction $\widetilde{u}_{l+k+1}$ in Case 2. For this purpose let us consider the optimal solution $\widehat{x}_i$ of (4.22), where we assume w.l.o.g. that $\widehat{x}_i$ is a vertex of P. By defining

$$\bar{u}_i := \widehat{x}_i - x_0'$$

we get a ray $x_0' + \tau\bar{u}_i$, $\tau \geq 0$, that contains the vertex $\widehat{x}_i$ of P. It holds $g_i^{\mathsf{T}}x_0' < g_i^{\mathsf{T}}\widehat{x}_i$ which implies

$$g_i^{\mathsf{T}}(x_0' + \tau\bar{u}_i) \begin{cases} < g_i^{\mathsf{T}}\widehat{x}_i & \text{for } 0 \leq \tau < 1, \\ \geq g_i^{\mathsf{T}}\widehat{x}_i & \text{for } \tau \geq 1. \end{cases} \tag{4.23}$$

Because of $P \subset \{x \in I\!R \mid g_i^{\mathsf{T}}x \leq g_i^{\mathsf{T}}\widehat{x}_i\}$ and (4.23) for $\tau \geq 1$, the ray $x_0' + \tau\bar{u}_i$ does not contain interior points of P. However, it may contain interior points of P for some τ with $0 < \tau < 1$. To check this we determine

$$\tau_i^* = \min\{x_0' + \tau\bar{u}_i \in P \mid \tau \geq 0\}. \tag{4.24}$$

Note that we always have $0 < \tau_i^* \leq 1$. If we have $\tau_i^* = 1$, then $x_0 + \tau\bar{u}_i$, $\tau \geq 0$, contains no interior points of P and we can set $\widetilde{u}_{l+k+1} := \bar{u}_i$.

However, if $\tau_i^* < 1$, then the ray $x_0' + \tau\bar{u}_i$, $\tau \geq 0$, may contain interior points of P. Since $x_0' + \tau_i^*\bar{u}_i$ and $\widehat{x}_i$ lie on the boundary of P and P is convex, the ray $x_0' + \tau\bar{u}_i$, $\tau \geq 0$, contains *no* interior points if and only if

$$\frac{1}{2}\Big((x_0' + \tau_i^*\bar{u}_i) + \widehat{x}_i\Big) \notin \mathrm{int}(P). \tag{4.25}$$

If (4.25) holds, then, as above, we set $\widetilde{u}_{l+k+1} = \bar{u}_i$. Otherwise we have to determine another vertex of P lying above the facet $\mathcal{G}_i$ with which we can derive a cone edge fulfilling the respective conditions. This is done as follows.

First we set $g_i^{(0)} := g_i$, $\theta_i^{(0)} := \theta_i$, $\widehat{x}_i^{(0)} := \widehat{x}_i$ and $\bar{u}_i^{(0)} := \bar{u}_i$. Let us now consider the hyperplane $g_i^{(0)\mathsf{T}}x = \theta_i^{(0)}$ that contains x_0' and is spanned by the directions $\widetilde{u}_{i_1}, \widetilde{u}_{i_2}, \ldots, \widetilde{u}_{i_{n-1}}$ of $\widetilde{C}^{(k)}(x_0')$. We choose one of these directions, in our case, for instance, $\tilde{u}_{i_{n-1}}$, replace this direction by $\bar{u}_i^{(0)}$, and set $j := 0$. Then we determine the hyperplane $g_i^{(j+1)\mathsf{T}} x = \theta_i^{(j+1)}$ that contains x_0' and is spanned by the directions $\tilde{u}_{i_1}, \tilde{u}_{i_2}, \ldots, \tilde{u}_{i_{n-1}}, \bar{u}_i^{(j)}$. By solving the linear program

$$\max\{g_i^{(j+1)\mathsf{T}} x \mid x \in P\} \tag{4.26}$$

we get a vertex $\hat{x}_i^{(j+1)}$ of P with $g_i^{(j+1)\mathsf{T}} \hat{x}_i^{(j+1)} > \theta_i^{(j+1)}$. Note that the ray $x_0' + \tau \bar{u}_i^{(j)}$, $\tau \geq 0$, which is contained in the hyperplane $g_i^{(j+1)\mathsf{T}} x = \theta_i^{(j+1)}$, contains interior points of P. We modify the direction $\bar{u}_i^{(j)}$ by defining

$$\bar{u}_i^{(j+1)} := \hat{x}_i^{(j+1)} - x_0'.$$

If this modified ray $x_0' + \tau \bar{u}_i^{(j+1)}$, $\tau \geq 0$, contains no interior points of P, which can be determined as above, then we are done, i.e., we define $\tilde{u}_{l+k+1} = \bar{u}_i^{(j+1)}$. Otherwise we set $j := j + 1$ and repeat this procedure. It is not hard to verify that this procedure is finite.

This whole procedure for constructing the cone $\tilde{C}(x_0')$ in (4.19) stops after a finite number of iterations with a cone $\tilde{C}^{(k)}(x_0') = \text{cone}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_t)$ such that for all facets of this cone the conditions of Case 1 are fulfilled. Then we set $\tilde{C}(x_0') := \tilde{C}^{(k)}(x_0')$. In Figure 8 we have indicated such a cone, which was derived from the base cone $\hat{C}(x_0')$ depicted in Figure 7 in the fashion described above. It holds:

PROPOSITION 4.2 *Let $\tilde{C}(x_0') := \tilde{C}^{(k)}(x_0')$ denote the final cone of the iterative procedure described above. Then we have $C(x_0') = \tilde{C}(x_0')$, i.e. $\tilde{C}(x_0')$ is the smallest cone vertexed at x_0' that contains P.*

Proof. Each extreme ray $x_0' + \tau \tilde{u}_i$, $\tau \geq 0$ of the cone $\tilde{C}(x_0')$ touches, by construction, the boundary of P. Hence these rays are contained in $C(x_0')$. Since $C(x_0')$ is a convex cone, we therefore have

$$\tilde{C}(x_0') \subseteq C(x_0'). \tag{4.27}$$

But $\tilde{C}(x_0')$ is also a convex cone vertexed at x_0'. Furthermore, since for all its facets Case 1 holds, we have $P \subset \tilde{C}(x_0')$. However, $C(x_0')$ is, by definition, the smallest P-containing cone vertexed at x_0'. Hence we have $C(x_0') \subseteq \tilde{C}(x_0')$. With (4.27) we therefore have $C(x_0') = \tilde{C}(x_0')$, which proves the Proposition. □

According to Proposition 4.2 we have $C(x_0') = \tilde{C}(x_0')$. However, some directions of the recession cone of $\tilde{C}(x_0') = x_0' + \text{cone}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_t)$ are usually superfluous, i.e. the representation of the recession cone is not minimal. For instance, those directions derived in the construction of the base cone which contain no vertices of P can be eliminated. Therefore, to obtain a representation of the cone $C(x_0')$ in the form (4.16) we have to identify and eliminate all the superfluous directions. This can be done in the usual way. However, this final step is not absolutely necessary, since, as can easily be verified, deriving a valid cut w.r.t. the cone $C(x_0')$ or w.r.t. the cone $\tilde{C}(x_0')$ results in identical cuts.

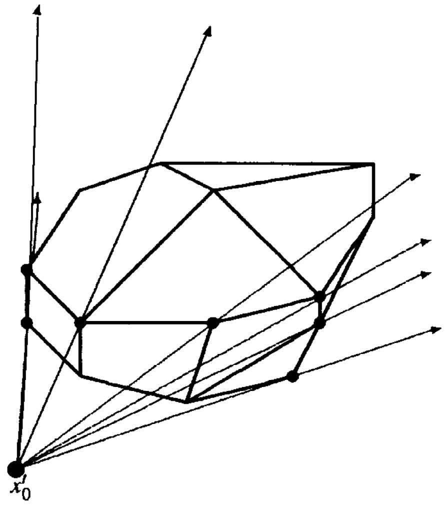

Figure 8. The cone $C(x_0')$

5. Iterative Cut Improvement

The starting point for our considerations in this paper was an x_0-eliminating concavity cut $c^{\mathsf{T}}(x - x_0) \geq 1$ that was derived w.r.t. the cone $C(x_0)$. As we have seen in the second section, we can get a deeper cut by deriving a cut w.r.t. the cone $C(x_0')$, where $x_0' = x_0 - \lambda_0 c \in \mathrm{bd}(L(\widehat{f} - \varepsilon))$ and $\lambda_0 > 0$. If $d^{\mathsf{T}}(x - x_0) \geq 1$ is the cut derived w.r.t. $C(x_0')$, then it is a valid cut which, in general, dominates the corresponding concavity cut. However, we can further improve the cut $d^{\mathsf{T}}(x - x_0) \geq 1$ in an iterative fashion by pulling the base of the cone further in the direction $-c$. This is based on the following considerations.

First, for $\lambda^+ \geq \lambda_0$ the only part of the cone $C(x_0(\lambda^+))$ with $x_0(\lambda^+) = x_0 - \lambda^+ c$ that is of interest to us is the part which lies in the half-space $d^{\mathsf{T}}(x - x_0) \geq 1$. This follows from our knowledge that the part lying in the half-space $d^{\mathsf{T}}(x - x_0) \leq 1$ does not contain a point $x \in P$ with $f(x) < \widehat{f} - \varepsilon$. Hence we can restrict ourselves to considering

$$C(x_0(\lambda^+)) \cap \{x \in I\!R^n \mid d^{\mathsf{T}}(x - x_0) \geq 1\}.$$

Let $C(x_0(\lambda^+))$ be of the form

$$C(x_0(\lambda^+)) = x_0(\lambda^+) + \mathrm{cone}\Big(u_1(\lambda^+), u_2(\lambda^+), \ldots, u_{s_{\lambda^+}}(\lambda^+)\Big),$$

and for $i = 1, 2, \ldots, s_{\lambda^+}$ let $\bar{x}_i(\lambda^+)$ denote the intersection point of the edge $E_{i,\lambda^+}(\tau) = x_0(\lambda^+) + \tau u_i(\lambda^+)$, $\tau \geq 0$, with the hyperplane $d^{\mathsf{T}}(x - x_0) = 1$. Then

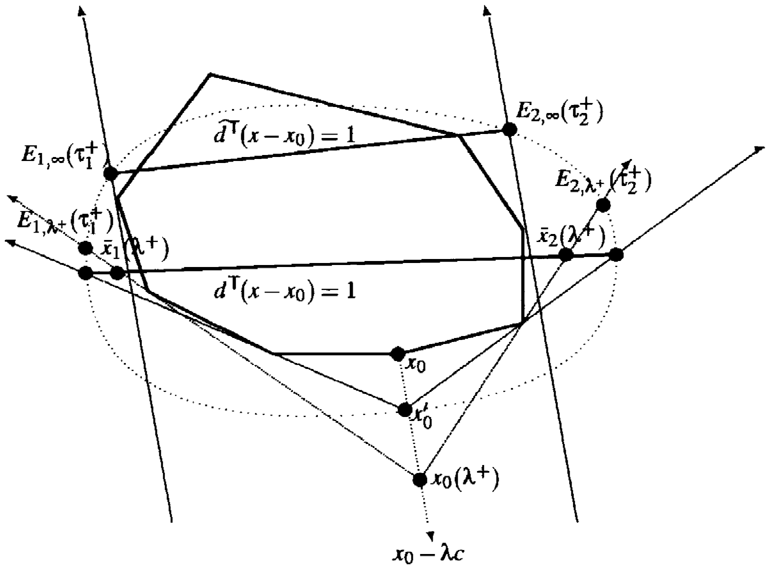

Figure 9. Cut improvement with positive values of λ

we have

$$
\begin{aligned}
&C(x_0(\lambda^+)) \cap \{x \in I\!R^n \mid d^{\mathsf{T}}(x-x_0) \geq 1\} \\
&= \operatorname{conv}\Big(\bar{x}_1(\lambda^+), \dots, \bar{x}_{s_{\lambda^+}}(\lambda^+)\Big) + \operatorname{cone}\Big(u_1(\lambda^+), \dots, u_{s_{\lambda^+}}(\lambda^+)\Big).
\end{aligned}
\tag{5.28}
$$

Second, by construction of the cut $d^{\mathsf{T}}(x-x_0) \geq 1$ we have $\bar{x}_i(\lambda^+) \in L(\widehat{f}-\varepsilon)$ for $i = 1, 2, \dots, s_{\lambda^+}$ and $\lambda^+ = \lambda_0$. In general, if we increase the value of λ^+ the distance from $\bar{x}_i(\lambda^+)$ to the boundary of $L(\widehat{f}-\varepsilon)$ also increases. This is illustrated in Figure 9.

By choosing an appropriate λ^+ with $\lambda^+ > \lambda_0$ such that

$$
\bar{x}_i(\lambda^+) \in \operatorname{int}(L(\widehat{f}-\varepsilon)) \qquad \text{for } i = 1, 2, \dots, s_{\lambda^+}
$$

we can make use of (4.28) to derive a cut that dominates the cut $d^{\mathsf{T}}(x-x_0) \geq 1$. For this purpose we determine the maximal τ_i^+ such that $E_{i,\lambda^+}(\tau_i^+)$ lies on the boundary of $L(\widehat{f}-\varepsilon)$ and, in analogy to (4.15), solve the linear program

$$
\begin{array}{lrll}
\text{minimize} & c^{\mathsf{T}}d & & \\
\text{s.t.} & d^{\mathsf{T}}(E_{i,\lambda^+}(\tau_i^+)-x_0) & \geq 1 & \text{for } i = 1, 2, \dots, s_{\lambda^+}, \\
& d^{\mathsf{T}}(\bar{x}_i(\lambda^+)-x_0) & \leq 1 & \text{for } i = 1, 2, \dots, s_{\lambda^+}.
\end{array}
\tag{5.29}
$$

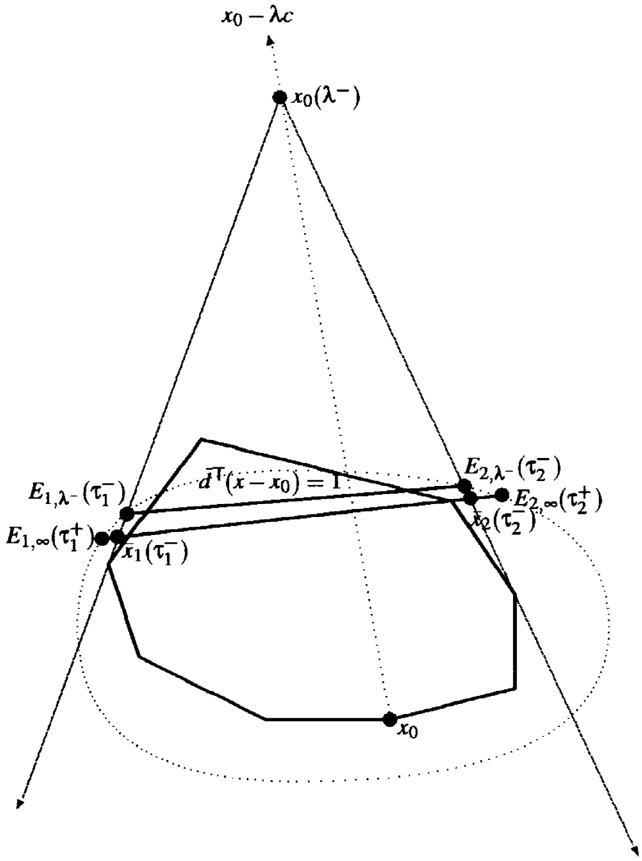

Figure 10. Cut improvement with negative values of λ

Let $\widehat{d}$ be an optimal solution of (4.29). Then $\widehat{d}^{\mathsf{T}}(x - x_0) \geq 1$ is a valid cut for $P \cap \{x \in I\!R^n \mid d^{\mathsf{T}}(x - x_0) \geq 1\}$. This follows from

$$\begin{aligned}
&\{x \in P \mid d^{\mathsf{T}}(x - x_0) \geq 1\} \cap \{x \in I\!R^n \mid \widehat{d}^{\mathsf{T}}(x - x_0) \leq 1\} \\
\subset\ & C(x_0(\lambda^+)) \cap \{x \in I\!R^n \mid d^{\mathsf{T}}(x - x_0) \geq 1,\ \widehat{d}^{\mathsf{T}}(x - x_0) \leq 1\} \\
\subset\ & \operatorname{conv}\Big(\bar{x}_1(\lambda^+), \dots, \bar{x}_{s_{\lambda^+}}(\lambda^+), E_{1,\lambda^+}(\tau_1^+), \dots, E_{s_{\lambda^+},\lambda^+}(\tau_{s_{\lambda^+}}^+)\Big) \\
\subset\ & L(\widehat{f} - \varepsilon).
\end{aligned}$$

This means that the cut $\widehat{d}^{\mathsf{T}}(x - x_0) \geq 1$ eliminates no x in $P \cap \{x \in I\!R^n \mid d^{\mathsf{T}}(x - x_0) \geq 1\}$ with $f(x) < \widehat{f} - \varepsilon$. However, by construction the cut $d^{\mathsf{T}}(x - x_0) \geq 1$ is either dominated by or equivalent to the cut $\widehat{d}^{\mathsf{T}}(x - x_0) \geq 1$. Furthermore, the cut $d^{\mathsf{T}}(x - x_0) \geq 1$ is a valid cut for P. This implies that $\widehat{d}^{\mathsf{T}}(x - x_0) \geq 1$ is also a valid cut for P. This can also be seen in Figure 9, which shows such a valid cut $\widehat{d}^{\mathsf{T}}(x - x_0) \geq 1$.

We can repeat this procedure with the cone $C(x_0(\lambda^+))$ and the cut $\widehat{d}^{\mathsf{T}}(x - x_0) \geq 1$, and so on. In this way we get an iterative cut improvement procedure.

In an extreme situation of the iterative procedure we may end up with a cone $C(x_0(\lambda))$ with $\lambda = \infty$. In this case the edges $E_{i,\lambda}(\tau)$ are parallel to the line $x_0 - \lambda c$, $\lambda \in I\!R$. Such a situation and the corresponding cut are also illustrated in Figure 9. A comparison of this cut with the initial cut depicted in Figure 5 shows that the procedure for cut improvement can lead to substantially deeper cuts. Even more dramatic is the improvement in comparison to the concavity cut of Figure 1.

However, even for $\lambda = \infty$ some further improvement of the resulting cut may be possible. The basic idea here is to choose a negative value λ^- for λ. Starting with a very small (negative) value for λ we successively increase λ. As can be seen in Figure 10, the corresponding cones and the concavity cut cone $C(x_0)$ first shown in Figure 1 lie face to face.

As for positive values of λ we have to choose λ^- with $\lambda^- < 0$ in such a way that the edges of the cone $C(x_0(\lambda^-))$ intersect the cut which was derived in the previous iteration in the interior of $L(\widehat{f} - \varepsilon)$. We can derive a cutting plane w.r.t. $C(x_0(\lambda^-))$ by a linear program similar to (4.29). Note that in this case $E_{i,\lambda^-}(\tau_i^-)$ is the point where $E_{i,\lambda^+}(\tau_i)$ intersects the boundary of $L(\widehat{f} - \varepsilon)$ for the *first* time. Such a cut $\bar{d}^{\mathsf{T}}(x - x_0) = 1$, which dominates the cut derived for $\lambda = \infty$, is indicated in Figure 10.

6. Concluding Remarks

Concavity cuts were first introduced in the context of concave minimization but they since also found application in different types of algorithms for other global optimization problems, e.g., reverse convex programming, bilinear programming and Lipschitzian optimization. It is still unclear, however, whether the finite convergence of a cutting plane algorithm for concave minimization can be ensured by concavity cuts or not.

The main goal of the present paper was to modify concavity cuts in such a way that the finite convergence of a corresponding cutting plane algorithm can be established without introducing enumerative elements such as facial cuts. The basic idea behind the proposed modification is simple: just pull the base of the cone that is used to derive a Tuy cut away from the polytope P.

However, determining an explicit representation of the resulting cone can be quite time-consuming. Hence deriving a cut in the manner proposed in this paper may be much more expensive than deriving a concavity cut. But as experiments with cutting planes derived by cone decomposition have shown (cf., Porembski, 1999), it may be worth the added expenses if the derived cut is much deeper than the corresponding concavity cut. The cost of deriving such a cut may be high, but the number of cuts needed to solve the concave minimization problem may be much smaller than with concavity cuts. However, this remains to be thoroughly examined in computational experiments.

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